Contents 7

Matrices

Learning outcomes

In this Workbook you will learn about matrices. In the first instance you will learn about the algebra of matrices: how they can be added, subtracted and multiplied. You will learn about a characteristic quantity associated with square matrices - the determinant. Using knowledge of determinants you will learn how to find the inverse of a matrix. Also, a second method for finding a matrix inverse will be outlined - the Gaussian elimination method.

A working knowledge of matrices is a vital attribute of any mathematician, engineer or scientist. You will find that matrices arise in many varied areas of science.

Introduction to Matrices

7.1

Introduction

When we wish to solve large systems of simultaneous linear equations, which arise for example in the problem of finding the forces on members of a large framed structure, we can isolate the coefficients of the variables as a block of numbers called a matrix. There are many other applications matrices. In this Section we develop the terminology and basic properties of a matrix.

Prerequisites

Before starting this Section you should ...

- be familiar with the rules of number algebra

Learning Outcomes

On completion you should be able to ...

- express a system of linear equations in matrix form
- recognise and use the basic terminology associated with matrices
- carry out addition and subtraction with two given matrices or state that the operation is not possible

1. Applications of matrices

The solution of simultaneous linear equations is a task frequently occurring in engineering. In electrical engineering the analysis of circuits provides a ready example.
However the simultaneous equations arise, we need to study two things:

(a) how we can conveniently represent large systems of linear equations

(b) how we might find the solution of such equations.

We shall discover that knowledge of the theory of matrices is an essential mathematical tool in this area.

Representing simultaneous linear equations

Suppose that we wish to solve the following three equations in three unknowns x_1, x_2 and x_3:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 3 \\ x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 + 4x_3 &= 5 \end{aligned}$$

We can isolate three facets of this system: the **coefficients** of x_1, x_2, x_3; the **unknowns** x_1, x_2, x_3; and the **numbers** on the right-hand sides.
Notice that in the system

$$\begin{aligned} 3x + 2y - z &= 3 \\ x - y + z &= 4 \\ 2x + 3y + 4z &= 5 \end{aligned}$$

the only difference from the first system is the names given to the unknowns. It can be checked that the first system has the solution $x_1 = 2,\ x_2 = -1,\ x_3 = 1$. The second system therefore has the solution $x = 2, y = -1, z = 1$.
We can isolate the three facets of the first system by using **arrays** of numbers and of unknowns:

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Even more conveniently we represent the arrays with letters (usually capital letters)

$$AX = B$$

Here, to be explicit, we write

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Here A is called the **matrix of coefficients**, X is called the **matrix of unknowns** and B is called the **matrix of constants**.
If we now append to A the column of right-hand sides we obtain the **augmented matrix** for the system:

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & 3 \\ 1 & -1 & 1 & 4 \\ 2 & 3 & 4 & 5 \end{array}\right]$$

The order of the entries, or elements, is crucial. For example, all the entries in the second row relate to the second equation, the entries in column 1 are the coefficients of the unknown x_1, and those in the last column are the constants on the right-hand sides of the equations.
In particular, the entry in row 2 column 3 is the coefficient of x_3 in equation 2.

Representing networks

Shortest-distance problems are important in communications study. Figure 1 illustrates schematically a system of four towns connected by a set of roads.

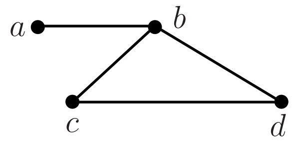


Figure 1

The system can be represented by the matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

The row refers to the town from which the road starts and the column refers to the town where the road ends. An entry of 1 indicates that two towns are directly connected by a road (for example b and d) and an entry of zero indicates that there is no direct road (for example a and c). Of course, if there is a road from b to d (say) it is also a road from d to b.

In this Section we shall develop some basic ideas about matrices.

2. Definitions

An array of numbers, rectangular in shape, is called a **matrix**. The first matrix below has 3 rows and 2 columns and is said to be a '3 by 2' matrix (written 3×2). The second matrix is a '2 by 4' matrix (written 2×4).

$$\left[\begin{array}{rr} 1 & 4 \\ -2 & 3 \\ 2 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{array}\right]$$

The general 3×3 matrix can be written

$$A = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]$$

where a_{ij} denotes the element in row i, column j.
For example in the matrix:

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 6 & -12 \\ 5 & 7 & 123 \end{bmatrix}$$

$$a_{11} = 0, \quad a_{12} = -1, \quad a_{13} = -3, \quad \ldots \quad a_{22} = 6, \quad \ldots \quad a_{32} = 7, \quad a_{33} = 123$$

Key Point 1

The General Matrix

A general $m \times n$ matrix A has m rows and n columns.

The entries in the matrix A are called the **elements** of A.

In matrix A the element in row i and column j is denoted by a_{ij}.

A matrix with only one column is called a **column vector** (or **column matrix**).

For example, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ are both 3×1 column vectors.

A matrix with only one row is called a **row vector** (or **row matrix**). For example $[2, -3, 8, 9]$ is a 1×4 row vector. Often the entries in a row vector are separated by commas for clarity.

Square matrices

When the number of rows is the same as the number of columns, i.e. $m = n$, the matrix is said to be **square** and of **order** n (or m).

- In an $n \times n$ square matrix A, the **leading diagonal** (or **principal diagonal**) is the 'north-west to south-east' collection of elements $a_{11}, a_{22}, \ldots, a_{nn}$. The sum of the elements in the leading diagonal of A is called the **trace** of the matrix, denoted by tr(A).

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \qquad \text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

- A square matrix in which all the elements below the leading diagonal are zero is called an **upper triangular matrix**, often denoted by U.

$$U = \begin{bmatrix} u_{11} & u_{12} & \ldots & \ldots & u_{1n} \\ 0 & u_{22} & \ldots & \ldots & u_{2n} \\ 0 & 0 & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & u_{nn} \end{bmatrix} \qquad u_{ij} = 0 \quad \text{when} \quad i > j$$

- A square matrix in which all the elements above the leading diagonal are zero is called a **lower triangular matrix**, often denoted by L.

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \ldots & 0 \\ l_{21} & l_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \ldots & 0 \\ l_{n1} & l_{n2} & \vdots & \ldots & l_{nn} \end{bmatrix} \qquad l_{ij} = 0 \quad \text{when} \quad i < j$$

- A square matrix where all the non-zero elements are along the leading diagonal is called a diagonal matrix, often denoted by D.

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \ldots & 0 \\ 0 & d_{22} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & d_{nn} \end{bmatrix} \qquad d_{ij} = 0 \quad \text{when} \quad i \neq j$$

Some examples of matrices and their classification

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is 2×3. It is not square.

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is 2×2. It is square.

Also, tr(A) does not exist, and tr$(B) = 1 + 4 = 5$.

$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ are both 3×3, square and upper triangular.

Also, tr$(C) = 0$ and tr$(D) = 3$.

$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -5 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ are both 3×3, square and lower triangular.

Also, tr$(E) = 0$ and tr$(F) = 4$.

$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and $H = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are both 3×3, square and diagonal.

Also, tr$(G) = 0$ and tr$(H) = 6$.

Task

Classify the following matrices (and, where possible, find the trace):

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -3 & -2 & -4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Your solution

Answer
A is 3×2, B is 3×4, C is 4×4 and square.

The trace is not defined for A or B. However, tr$(C) = 34$.

Task

Classify the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Your solution

Answer
A is 3×3 and square, B is 3×3 lower triangular, C is 3×3 upper triangular and D is 3×3 diagonal.

Equality of matrices

As we noted earlier, the terms in a matrix are called the **elements** of the matrix.

The elements of the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$ are $1, 2, -1, -4$

We say two matrices A, B are **equal** to each other only if A and B have the same number of rows and the same number of columns and if each element of A is equal to the corresponding element of B. When this is the case we write $A = B$. For example if the following two matrices are equal:

$$A = \begin{bmatrix} 1 & \alpha \\ -1 & -\beta \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$$

then we can conclude that $\alpha = 2$ and $\beta = 4$.

The unit matrix

The **unit matrix** or the **identity matrix**, denoted by I_n (or, often, simply I), is the diagonal matrix of order n in which all diagonal elements are 1.

Hence, for example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The zero matrix

The **zero matrix** or **null matrix** is the matrix all of whose elements are zero. There is a zero matrix for every size. For example the 2×3 and 2×2 cases are:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Zero matrices, of whatever size, are denoted by $\underline{0}$.

The transpose of a matrix

The **transpose** of a matrix A is a matrix where the rows of A become the columns of the new matrix and the columns of A become its rows. For example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The resulting matrix is called the **transposed matrix** of A and denoted A^T. In the previous example it is clear that A^T is not equal to A since the matrices are of different sizes. If A is square $n \times n$ then A^T will also be $n \times n$.

Example 1

Find the transpose of the matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution

Interchanging rows with columns we find

$$B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Both matrices are 3×3 but B and B^T are clearly different.

When the transpose of a matrix is equal to the original matrix i.e. $A^T = A$, then we say that the matrix A is **symmetric**. (This is because it has symmetry about the leading diagonal.)
In Example 1 B is **not** symmetric.

Example 2

Show that the matrix $C = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

Solution

Taking the transpose of C:

$$C^T = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}.$$

Clearly $C^T = C$ and so C is a symmetric matrix. Notice how the leading diagonal acts as a "mirror"; for example $c_{12} = -2$ and $c_{21} = -2$. In general $c_{ij} = c_{ji}$ for a symmetric matrix.

Task

Find the transpose of each of the following matrices. Which are symmetric?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Your solution

Answer

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = C,$ symmetric

$D^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} \quad E^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E,$ symmetric

3. Addition and subtraction of matrices

Under what circumstances can we add two matrices i.e. define $A + B$ for given matrices A, B?

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 5 & 6 & 9 \\ 7 & 8 & 10 \end{bmatrix}$$

There is no sensible way to define $A + B$ in this case since A and B are different sizes.
However, if we consider matrices of the same size then addition can be defined in a very natural way. Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. The 'natural' way to add A and B is to add corresponding elements together:

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

In general if A and B are both $m \times n$ matrices, with elements a_{ij} and b_{ij} respectively, then their sum is a matrix C, also $m \times n$, such that the elements of C are

$$c_{ij} = a_{ij} + b_{ij} \qquad i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, n$$

In the above example

$$c_{11} = a_{11} + b_{11} = 1 + 5 = 6 \qquad c_{21} = a_{21} + b_{21} = 3 + 7 = 10 \quad \text{and so on.}$$

Subtraction of matrices follows along similar lines:

$$D = A - B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

4. Multiplication of a matrix by a number

There is also a natural way of defining the product of a matrix with a number. Using the matrix A above, we note that

$$A + A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

What we see is that $2A$ (which is the shorthand notation for $A + A$) is obtained by multiplying *every* element of A by 2.
In general if A is an $m \times n$ matrix with typical element a_{ij} then the product of a number k with A is written kA and has the corresponding elements ka_{ij}.

Hence, again using the matrix A above,

$$7A = 7\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}$$

Similarly:

$$-3A = \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix}$$

Task

For the following matrices find, where possible, $A+B,\ A-B,\ B-A,\ 2A$.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Your solution

Answer

1. $A+B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad A-B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \qquad B-A = \begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix} \qquad 2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

2. $A+B = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 8 & 9 & 10 \end{bmatrix} \qquad A-B = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{bmatrix} \qquad B-A = \begin{bmatrix} 0 & -1 & -2 \\ -5 & -6 & -7 \\ -6 & -7 & -8 \end{bmatrix}$

   $2A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$

3. None of $A+B,\ A-B,\ B-A$, are defined. $\qquad 2A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$

5. Some simple matrix properties

Using the definition of matrix addition described above we can easily verify the following properties of matrix addition:

Key Point 2

Basic Properties of Matrices

Matrix addition is **commutative**: $A + B = B + A$

Matrix addition is **associative**: $A + (B + C) = (A + B) + C$

The **distributive law** holds: $k(A + B) = k\,A + k\,B$

These Key Point results follow from the fact that $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ etc.

We can also show that the transpose of a matrix satisfies the following simple properties:

Key Point 3

Properties of Transposed Matrices

$$\begin{aligned}(A+B)^T &= A^T + B^T \\ (A-B)^T &= A^T - B^T \\ (A^T)^T &= A\end{aligned}$$

Example 3

Show that $(A^T)^T = A$ for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Solution

$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ so that $(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$

Task

For the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ verify that

(i) $3(A+B) = 3A + 3B$ (ii) $(A-B)^T = A^T - B^T$.

Your solution

Answer

(i) $A + B = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}; \quad 3(A+B) = \begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}; \quad 3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix};$

$3B = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}; \quad 3A + 3B = \begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}.$

(ii) $A - B = \begin{bmatrix} 0 & 3 \\ 4 & 3 \end{bmatrix}; \quad (A-B)^T = \begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix};$

$B^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad A^T - B^T = \begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}.$

Exercises

1. Find the coefficient matrix A of the system:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ 4x_1 + 4x_2 &= 0 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

If $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ determine $(3A^T - B)^T$.

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 2 & 7 \end{bmatrix}$ verify that $3(A^T - B) = (3A - 3B^T)^T$.

Answers

1. $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & 0 \\ 2 & -1 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 4 & -1 \\ -1 & 0 & -1 \end{bmatrix}, \quad 3A^T = \begin{bmatrix} 6 & 12 & 6 \\ 9 & 12 & -3 \\ -3 & 0 & -3 \end{bmatrix}$

$3A^T - B = \begin{bmatrix} 5 & 10 & 3 \\ 5 & 7 & -9 \\ -3 & 0 & -4 \end{bmatrix} \quad (3A^T - B)^T = \begin{bmatrix} 5 & 5 & -3 \\ 10 & 7 & 0 \\ 3 & -9 & -4 \end{bmatrix}$

2. $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad A^T - B = \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}, \quad 3(A^T - B) = \begin{bmatrix} 6 & 0 \\ 6 & 12 \\ 3 & -3 \end{bmatrix}$

$B^T = \begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & 7 \end{bmatrix}, \quad 3A - 3B^T = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 6 \\ 12 & 3 & 21 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 3 \\ 0 & 12 & -3 \end{bmatrix}$

Matrix Multiplication 7.2

Introduction

When we wish to multiply matrices together we have to ensure that the operation is possible - and this is not always so. Also, unlike number arithmetic and algebra, even when the product exists the order of multiplication may have an effect on the result. In this Section we pick our way through the minefield of matrix multiplication.

Prerequisites

Before starting this Section you should ...

- understand the concept of a matrix and associated terms.

Learning Outcomes

On completion you should be able to ...

- decide when the product AB exists
- recognise that $AB \neq BA$ in most cases
- carry out the multiplication AB
- explain what is meant by the identity matrix I

1. Multiplying row matrices and column matrices together

Let A be a 1×2 row matrix and B be a 2×1 column matrix:

$$A = \begin{bmatrix} a & b \end{bmatrix} \qquad B = \begin{bmatrix} c \\ d \end{bmatrix}$$

The product of these two matrices is written AB and is the 1×1 matrix defined by:

$$AB = \begin{bmatrix} a & b \end{bmatrix} \times \begin{bmatrix} c \\ d \end{bmatrix} = [ac + bd]$$

Note that corresponding elements are multiplied together and the results are then added together. For example

$$\begin{bmatrix} 2 & -3 \end{bmatrix} \times \begin{bmatrix} 6 \\ 5 \end{bmatrix} = [12 - 15] = [-3]$$

This matrix product is easily generalised to other row and column matrices. For example if C is a 1×4 row matrix and D is a 4×1 column matrix:

$$C = \begin{bmatrix} 2 & -4 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix}$$

then we define the product of C with D as

$$CD = \begin{bmatrix} 2 & -4 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix} = [6 - 12 - 6 + 10] = [-2]$$

The only requirement is that the number of elements of the row matrix is the same as the number of elements of the column matrix.

2. Multiplying two 2×2 matrices

If A and B are two matrices then the product AB is obtained by multiplying the rows of A with the columns of B in the manner described above. This will only be possible if the number of elements in the rows of A is the same as the number of elements in the columns of B. In particular, we define the product of two 2×2 matrices A and B to be another 2×2 matrix C whose elements are calculated according to the following pattern

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{B}{\begin{bmatrix} w & x \\ y & z \end{bmatrix}} \underset{=}{=} \underset{C}{\begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}}$$

The rule for calculating the elements of C is described in the following Key Point:

Key Point 4

Matrix Product

$$AB = C$$

The element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

We illustrate this construction for the abstract matrices A and B given above:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} [\,a \;\; b\,]\begin{bmatrix} w \\ y \end{bmatrix} & [\,a \;\; b\,]\begin{bmatrix} x \\ z \end{bmatrix} \\ \\ [\,c \;\; d\,]\begin{bmatrix} w \\ y \end{bmatrix} & [\,c \;\; d\,]\begin{bmatrix} x \\ z \end{bmatrix} \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

For example

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} [\,2 \;\; -1\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,2 \;\; -1\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ \\ [\,3 \;\; -2\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,3 \;\; -2\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -6 & 10 \end{bmatrix}$$

Task

Find the product AB where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

First write down row 1 of A, column 2 of B and form the first element in product AB:

Your solution

Answer

$[1, 2]$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$; their product is $1 \times (-1) + 2 \times 1 = 1$.

Now repeat the process for row 2 of A, column 1 of B:

Your solution

Answer

$[3, 4]$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Their product is $3 \times 1 + 4 \times (-2) = -5$

Finally find the two other elements of $C = AB$ and hence write down the matrix C:

Your solution

Answer

Row 1 column 1 is $1 \times 1 + 2 \times (-2) = -3$. Row 2 column 2 is $3 \times (-1) + 4 \times 1 = 1$

$C = \begin{bmatrix} -3 & 1 \\ -5 & 1 \end{bmatrix}$

Clearly, matrix multiplication is tricky and not at all 'natural'. However, it is a very important mathematical procedure with many engineering applications so must be mastered.

3. Some surprising results

We have already calculated the product AB where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

Now complete the following task in which you are asked to determine the product BA, i.e. with the matrices in reverse order.

Task

For matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ form the products of

row 1 of B and column 1 of A row 1 of B and column 2 of A

row 2 of B and column 1 of A row 2 of B and column 2 of A

Now write down the matrix BA:

Your solution

Answer

row 1, column 1 is $1 \times 1 + (-1) \times 3 = -2$ row 1, column 2 is $1 \times 2 + (-1) \times 4 = -2$

row 2, column 1 is $-2 \times 1 + 1 \times 3 = 1$ row 2, column 2 is $-2 \times 2 + 1 \times 4 = 0$

BA is $\begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$

It is clear that AB and BA are **not** in general the same. In fact it is the **exception** that $AB = BA$. In the special case in which $AB = BA$ we say that the matrices A and B **commute**.

Task

Calculate AB and BA where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Your solution

Answer

$AB = BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

We call B the 2×2 **zero matrix** written $\underline{0}$ so that $A \times \underline{0} = \underline{0} \times A = 0$ for any matrix A.

Now in the multiplication of numbers, the equation

$$ab = 0$$

implies that either a is zero or b is zero or both are zero. The following task shows that this is not necessarily true for matrices.

Task

Carry out the multiplication AB where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Your solution

Answer

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Here we have a zero product yet neither A nor B is the zero matrix! Thus the statement $AB = 0$ does **not** allow us to conclude that either $A = \underline{0}$ or $B = \underline{0}$.

Task

Find the product AB where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Your solution

Answer

$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **identity matrix** or **unit matrix** of order 2, and is usually denoted by the symbol I. (Strictly we should write I_2, to indicate the size.) I plays the same role in matrix multiplication as the number 1 does in number multiplication.

Hence

just as $a \times 1 = 1 \times a = a$ for any number a, so $AI = IA = A$ for any matrix A.

4. Multiplying two 3×3 matrices

The definition of the product $C = AB$ where A and B are two 3×3 matrices is as follows

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} ar+bu+cx & as+bv+cy & at+bw+cz \\ dr+eu+fx & ds+ev+fy & dt+ew+fz \\ gr+hu+ix & gs+hv+iy & gt+hw+iz \end{bmatrix}$$

This looks a rather daunting amount of algebra but in fact the construction of the matrix on the right-hand side is straightforward if we follow the simple rule from Key Point 4 that the element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

For example, to obtain the element in row 2, column 3 of C we take row 2 of A: $[d, e, f]$ and multiply it with column 3 of B in the usual way to produce $[dt + ew + fz]$.

By repeating this process we obtain every element of C.

Task

Calculate $AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

First find the element in row 2 column 1 of the product:

Your solution

Answer

Row 2 of A is $(3, 4, 0)$ column 1 of B is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

The combination required is $3 \times 2 + 4 \times 1 + (0) \times (0) = 10$.

Now complete the multiplication to find all the elements of the matrix AB:

Your solution

Answer

In full detail, the elements of AB are:

$$\begin{bmatrix} 1\times 2+2\times 1+(-1)\times 0 & 1\times(-1)+2\times(-2)+(-1)\times 3 & 1\times 3+2\times 1+(-1)\times(-2) \\ 3\times 2+4\times 1+0\times 0 & 3\times(-1)+4\times(-2)+0\times 3 & 3\times 3+4\times 1+0\times(-2) \\ 1\times 2+5\times 1+(-2)\times 0 & 1\times(-1)+5\times(-2)+(-2)\times 3 & 1\times 3+5\times 1+(-2)\times(-2) \end{bmatrix}$$

i.e. $AB = \begin{bmatrix} 4 & -8 & 7 \\ 10 & -11 & 13 \\ 7 & -17 & 12 \end{bmatrix}$

The 3×3 unit matrix is: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and as in the 2×2 case this has the property that $AI = IA = A$

The 3×3 zero matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. Multiplying non-square matrices together

So far, we have just looked at multiplying 2×2 matrices and 3×3 matrices. However, products between non-square matrices may be possible.

Key Point 5

General Matrix Products

The general rule is that an $n \times p$ matrix A can be multiplied by a $p \times m$ matrix B to form an $n \times m$ matrix $AB = C$.

In words:

For the matrix product AB to be defined the number of columns of A must equal the number of rows of B.

The elements of C are found in the usual way:

The element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

Example 4

Find the product AB if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix}$

Solution

Since A is a 2×3 and B is a 3×2 matrix the product AB can be found and results in a 2×2 matrix.

$$AB = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 22 & 13 \\ 38 & 25 \end{bmatrix}$$

Task

Obtain the product AB if $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix}$

Your solution

Answer

AB is a 2×3 matrix.

$$AB = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} [\,1 \quad -2\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,1 \quad -2\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} & [\,1 \quad -2\,]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ & & \\ [\,2 \quad -3\,]\begin{bmatrix} 2 \\ 6 \end{bmatrix} & [\,2 \quad -3\,]\begin{bmatrix} 4 \\ 1 \end{bmatrix} & [\,2 \quad -3\,]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 1 \\ -14 & 5 & 2 \end{bmatrix}$$

6. The rules of matrix multiplication

It is worth noting that the process of multiplication can be continued to form products of more than two matrices.

Although two matrices may not commute (i.e. in general $AB \neq BA$) the **associative law** always holds i.e. for matrices which **can** be multiplied,

$$A(BC) = (AB)C.$$

The general principle is **keep the left to right order**, but within that limitation any two adjacent matrices can be multiplied.

It is important to note that it is not always possible to multiply together any two given matrices. For example if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ then $AB = \begin{bmatrix} a+2d & b+2e & c+2f \\ 3a+4d & 3b+4e & 3c+4f \end{bmatrix}$.

However $BA = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is **not defined** since each row of B has three elements whereas each column of A has two elements and we cannot multiply these elements in the manner described.

Given $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

State which of the products $AB,\ BA,\ AC,\ CA,\ BC,\ CB,\ (AB)C,\ A(CB)$ is defined and state the size $(n \times m)$ of the product when defined.

Your solution

AB
BA
AC
CA
BC
CB
$(AB)C$
$A(CB)$

Answer

$\begin{array}{cc} A & B \\ 2\times 3 & 2\times 2 \end{array}$	not possible	$\begin{array}{cc} B & A \\ 2\times 2 & 2\times 3 \end{array}$	possible; result 2×3
$\begin{array}{cc} A & C \\ 2\times 3 & 3\times 2 \end{array}$	possible; result 2×2	$\begin{array}{cc} C & A \\ 3\times 2 & 2\times 3 \end{array}$	possible; result 3×3
$\begin{array}{cc} B & C \\ 2\times 2 & 3\times 2 \end{array}$	not possible	$\begin{array}{cc} C & B \\ 3\times 2 & 2\times 2 \end{array}$	possible; result 3×2
$(AB)C$	not possible, AB not defined.	$\begin{array}{cc} A & (C\ B) \\ 2\times 3 & 3\times 2 \end{array}$	possible; result 2×2

We now list together some properties of matrix multiplication and compare them with corresponding properties for multiplication of numbers.

Key Point 6

Matrix algebra	Number algebra
$A(B+C) = AB + AC$	$a(b+c) = ab + ac$
$AB \neq BA$ in general	$ab = ba$
$A(BC) = (AB)C$	$a(bc) = (ab)c$
$AI = IA = A$	$1.a = a.1 = a$
$A\underline{0} = \underline{0}A = \underline{0}$	$0.a = a.0 = 0$
AB may not be possible	ab is always possible
$AB = \underline{0}$ does not imply $A = \underline{0}$ or $B = \underline{0}$	$ab = 0 \ \rightarrow \ a = 0$ or $b = 0$

Application of matrices to networks

A network is a collection of points (nodes) some of which are connected together by lines (paths). The information contained in a network can be conveniently stored in the form of a matrix.

Example 5

Petrol is delivered to terminals T_1 and T_2. They distribute the fuel to 3 storage depots ($S_1,\ S_2,\ S_3$). The network diagram below shows what fraction of the fuel goes from each terminal to the three storage depots. In turn the 3 depots supply fuel to 4 petrol stations (P_1, P_2, P_3, P_4) as shown in Figure 2:

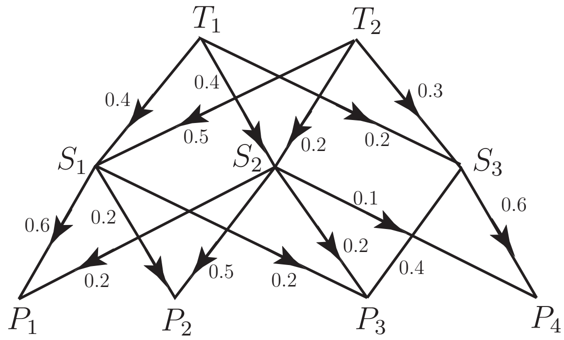
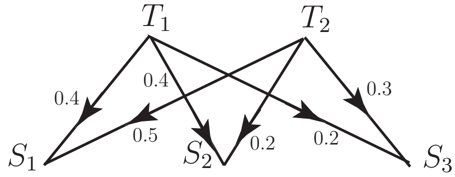


Figure 2

Show how this situation may be described using matrices.

Solution

Denote the amount of fuel, in litres, flowing from T_1 by t_1 and from T_2 by t_2 and the quantity being received at S_i by s_i for $i = 1, 2, 3$. This situation is described in the following diagram:

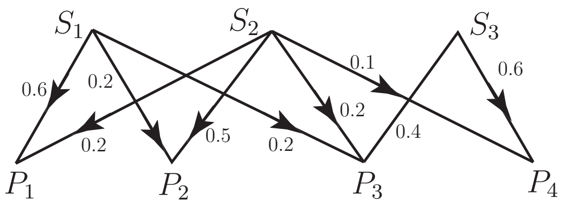


From this diagram we see that

$$\begin{aligned} s_1 &= 0.4t_1 + 0.5t_2 \\ s_2 &= 0.4t_1 + 0.2t_2 \\ s_3 &= 0.2t_1 + 0.3t_2 \end{aligned} \qquad \text{or, in matrix form:} \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

Solution (contd.)

In turn the 3 depots supply fuel to 4 petrol stations as shown in the next diagram:



If the petrol stations receive p_1, p_2, p_3, p_4 litres respectively then from the diagram we have:

$$\begin{aligned} p_1 &= 0.6s_1 + 0.2s_2 \\ p_2 &= 0.2s_1 + 0.5s_2 \\ p_3 &= 0.2s_1 + 0.2s_2 + 0.4s_3 \\ p_4 &= 0.1s_2 + 0.6s_3 \end{aligned} \quad \text{or, in matrix form:} \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Combining the equations, substituting expressions for s_1, s_2, s_3 in the equations for p_1, p_2, p_3, p_4 we get:

$$\begin{aligned} p_1 &= 0.6s_1 + 0.2s_2 \\ &= 0.6(0.4t_1 + 0.5t_2) + 0.2(0.4t_1 + 0.2t_1) \\ &= 0.32t_1 + 0.34t_2 \end{aligned}$$

with similar results for p_2, p_3 and p_4.

This is equivalent to combining the two networks. The results can be obtained more easily by multiplying the matrices:

$$\begin{aligned} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} &= \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\ &= \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.5 & 0 \\ 0.2 & 0.2 & 0.4 \\ 0 & 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \\ &= \begin{bmatrix} 0.32 & 0.34 \\ 0.28 & 0.20 \\ 0.24 & 0.26 \\ 0.16 & 0.20 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0.32t_1 + 0.34t_2 \\ 0.28t_1 + 0.20t_2 \\ 0.24t_1 + 0.26t_2 \\ 0.16t_1 + 0.20t_2 \end{bmatrix} \end{aligned}$$

Engineering Example 1

Communication network

Problem in words

Figure 3 represents a communication network. Vertices a, b, f and g represent offices. Vertices c, d and e represent switching centres. The numbers marked along the edges represent the number of connections between any two vertices. Calculate the number of routes from a and b to f and g

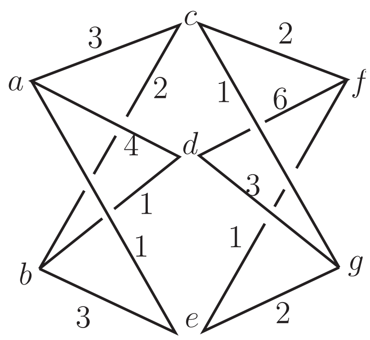


Figure 3: A communication network where a, b, f and g are offices and c, d and e are switching centres

Mathematical statement of the problem

The number of routes from a to f can be calculated by taking the number via c plus the number via d plus the number via e. In each case this is given by multiplying the number of connections along the edges connecting a to c, c to f etc. This gives the result:
Number of routes from a to $f = 3 \times 2 + 4 \times 6 + 1 \times 1 = 31$.

The nature of matrix multiplication means that the number of routes is obtained by multiplying the matrix representing the number of connections from ab to cde by the matrix representing the number of connections from cde to fg.

Mathematical analysis

The matrix representing the number of routes from ab to cde is:

$$\begin{array}{cc} & \begin{array}{ccc} c & d & e \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 3 \end{pmatrix} \end{array}$$

The matrix representing the number of routes from cde to fg is:

$$\begin{array}{cc} & \begin{array}{cc} f & g \end{array} \\ \begin{array}{c} c \\ d \\ e \end{array} & \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ 1 & 2 \end{pmatrix} \end{array}$$

The product of these two matrices gives the total number of routes.

$$\begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3\times 2+4\times 6+1\times 1 & 3\times 1+4\times 3+1\times 2 \\ 2\times 2+1\times 6+3\times 1 & 2\times 1+1\times 3+3\times 2 \end{pmatrix} = \begin{pmatrix} 31 & 17 \\ 13 & 11 \end{pmatrix}$$

Interpretation

We can interpret the resulting (product) matrix by labelling the columns and rows.

$$\begin{array}{cc} & \begin{array}{cc} f & g \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 31 & 17 \\ 13 & 11 \end{pmatrix} \end{array}$$

Hence there are 31 routes from a to f, 17 from a to g, 13 from b to f and 11 from b to g.

Exercises

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $\quad C = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$ find

   (a) AB, (b) AC, (c) $(A+B)C$, (d) $AC+BC$ (e) $2A-3C$

2. If a rotation through an angle θ is represented by the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and a second rotation through an angle ϕ is represented by the matrix $B = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ show that both AB and BA represent a rotation through an angle $\theta + \phi$.

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -1 & 2 \\ 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find AB and BC.

4. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, $\quad C = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$,

   verify $A(BC) = (AB)C$.

5. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ then show that AA^T is symmetric.

6. If $A = \begin{bmatrix} 11 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ verify that $(AB)^T = \begin{bmatrix} 0 & 1 \\ 11 & 3 \\ 22 & 7 \end{bmatrix} = B^T A^T$

Answers

1. (a) $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$ (b) $AC = \begin{bmatrix} 4 & -7 \\ 8 & -15 \end{bmatrix}$ (c) $(A+B)C = \begin{bmatrix} 16 & -30 \\ 24 & -46 \end{bmatrix}$

   (d) $AC + BC = \begin{bmatrix} 16 & -30 \\ 24 & -46 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 7 \\ 0 & 17 \end{bmatrix}$

2. $$AB = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}$$

   $$= \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$

   which clearly represents a rotation through angle $\theta + \phi$. BA gives the same result.

3. $AB = \begin{bmatrix} 15 & 26 \\ -6 & -12 \\ 12 & 24 \end{bmatrix}$, $\quad BC = \begin{bmatrix} 8 & 10 \\ 0 & 3 \\ 16 & 17 \end{bmatrix}$

4. $A(BC) = (AB)C = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$

Determinants

7.3

Introduction

Among other uses, determinants allow us to **determine** whether a system of linear equations has a unique solution or not. The evaluation of a determinant is a key skill in engineering mathematics and this Section concentrates on the evaluation of small size determinants. For evaluating larger sizes we can often use some properties of determinants to help simplify the task.

Prerequisites

Before starting this Section you should ...

- know what a matrix is

Learning Outcomes

On completion you should be able to ...

- evaluate a 2×2 determinant
- use the method of expansion along the top row to evaluate a determinant
- use the properties of determinants to aid their evaluation

1. Determinant of a 2×2 matrix

The **determinant** of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (note the change from square brackets to vertical lines) and is defined to be the number $ad - bc$. That is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We can use the notation det(A) or $\mid A \mid$ or Δ to denote the determinant of A.

Task

Find the determinants of the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -1 \\ -2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \qquad F = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}.$$

Your solution

Answer

$\mid A \mid = 1 \times 4 - 2 \times 3 = -2$ $\qquad \mid B \mid = 4 \times (-3) - (-1) \times (-2) = -12 - 2 = -14$

$\mid C \mid = 0$ $\qquad \mid D \mid = 3$ $\qquad \mid E \mid = 8$ $\qquad \mid F \mid = 3$ $\qquad \mid G \mid = -4 + 4 = 0$

2. Laplace expansion along the top row

This is a technique which can be used to evaluate determinants of any order. In principle, this method can use any row or any column as its starting point. We quote one example: using the top row.

Consider $\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$.

First we introduce the idea of a **minor**. Each element in this array of numbers has an associated minor formed by removing the column and row in which the element lies and taking the determinant of the remainder. For example consider element $a_{23} = 3$. We strike out the second row and the third column:

$$\begin{array}{ccc} 4 & 1 & \cancel{1} \\ \cancel{1} & \cancel{2} & \cancel{3} \\ 3 & 1 & \cancel{2} \end{array} \qquad \text{to leave} \qquad \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 4 - 3 = 1.$$

For the element $a_{31} = 3$ we strike out the third row and first column:

$$\begin{array}{ccc} \cancel{4} & 1 & 1 \\ \cancel{1} & 2 & 3 \\ \cancel{3} & \cancel{1} & \cancel{2} \end{array} \qquad \text{to leave} \qquad \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1.$$

Task

What is the minor of the element $a_{22} = 2$?

Your solution

Answer

$\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8 - 3 = 5$

Next we introduce the idea of a **cofactor**. This is a minor with a sign attached. The appropriate sign comes from the pattern of signs appropriate to a 3×3 array:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

(i.e. positive signs on the leading diagonal and the signs 'alternate' everywhere else.)
Each element has a cofactor associated with it. The cofactor of element a_{11} is denoted by A_{11}, that of a_{23} by A_{23} and so on.

To obtain the cofactor of an element of a 3×3 matrix we simply multiply the minor of that element by the corresponding sign from the 3×3 array of signs.
Hence the cofactor corresponding to a_{23} is

$$A_{23} = -\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = -1$$

and the cofactor corresponding to a_{31} is $A_{31} = +\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1.$

Task

What is the cofactor of the element a_{22}?

Your solution

Answer

The sign in the position of a_{22} in the array of signs is $+$

Hence, since the minor of this element is +5 the cofactor is $A_{22} = +5$.

Cofactors are important as it can be shown that the value of the determinant of a 3×3 matrix can be found from the formula

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

In words "the determinant of a 3×3 matrix is obtained by multiplying each element of the first row by its corresponding cofactor and then adding the three together". (In fact this rule can be extended to apply to **any** row or **any** column and to **any** order square matrix.)

Key Point 7

Evaluating General Determinants

If A is an $n \times n$ square matrix then : $\qquad \det(A) = \sum_{j=1}^{n} a_{ij}A_{ij}$

In words:

The determinant of a square matrix is obtained by multiplying each element of row i by its corresponding cofactor and then adding these products together.

In the case of $\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$ we have $a_{11} = 4,\ a_{12} = 1,\ a_{13} = 1,$

$$A_{11} = +\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2 - 9) = 7$$

$$A_{13} = +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5$$

Hence $\quad \Delta = 4 \times 1 \ + \ 1 \times 7 \ + \ 1 \times -5 \ = \ 6.$

Alternatively, choosing to expand along the second row:

$$\begin{aligned} \Delta &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= 1\left(-\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right) + 2\left(\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}\right) + 3\left(-\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}\right) = 6 \qquad \text{as before.} \end{aligned}$$

Task

Use expansion along the first row to find $\Delta = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -2 & 1 & 5 \end{vmatrix}$

Your solution

Answer

$a_{11} = 1, \quad a_{12} = -1, \quad a_{13} = 3$

$$\begin{aligned} A_{11} &= +\begin{vmatrix} 2 & 6 \\ 1 & 5 \end{vmatrix} = 10 - 6 = 4 \\ A_{12} &= -\begin{vmatrix} 0 & 6 \\ -2 & 5 \end{vmatrix} = -(0 + 12) = -12 \\ A_{13} &= +\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} = 2 + 2 = 4. \end{aligned}$$

Hence $\Delta = 1 \times 4 + (-1) \times (-12) + 3 \times 4 = 4 + 12 + 12 = 28$, as before.

3. Properties of determinants

Often, especially with determinants of large order, we can simplify the evaluation rules. In this Section we quote some useful properties of determinants in general.

1. If two rows (or two columns) of a determinant are interchanged then the value of the determinant is multiplied by (-1).

   For example $\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$ but (interchanging columns) $\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3 - 8 = -5$ and (interchanging rows) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5.$

2. The determinant of a matrix A and the determinant of its transpose A^T are equal.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$$

3. If two rows (or two columns) of a matrix A are equal then it has zero determinant.

   For example, the following determinant has two identical rows:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 1 \times \left(\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \right) + 2 \times \left(- \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \right) + 3 \times \left(\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \right)$$
$$= -3 + 2 \times (6) + 3 \times (-3) = 0.$$

4. If the elements of one row (or one column) of a determinant are multiplied by k, then the resulting determinant is k times the given determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 7 & 8 & 9 \end{vmatrix}.$$

   Note that if one row (or column) of a determinant is a multiple of another row (or column) then the value of the determinant is zero. (This follows from properties 3 and 4.)

   For example:

$$\begin{vmatrix} 2 & 4 & -1 \\ 4 & 2 & 1 \\ -4 & -8 & 2 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & 1 \\ -8 & 2 \end{vmatrix} + 4 \times \left(- \begin{vmatrix} 4 & 1 \\ -4 & 2 \end{vmatrix} \right) - 1 \times \begin{vmatrix} 4 & 2 \\ -4 & -8 \end{vmatrix}$$

$$= 2(12) + 4(-12) - (-24) = 0$$

   This is predictable as the 3rd row is (-2) times the first row.

5. If we add (or subtract) a multiple of one row (or column) to another, the value of the determinant is unchanged.

   Given $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$, add ($2 \times$ row 1) to (row 2) gives

$$\begin{vmatrix} 1 & 2 \\ 4 + 2 \times 1 & 5 + 2 \times 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 6 & 9 \end{vmatrix} = 9 - 12 = -3 = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

6. The determinant of a lower triangular matrix, an upper triangular matrix or a diagonal matrix is the product of the elements on the leading diagonal.

   As an example, it is easily confirmed that each of the following determinants has the same value $1 \times 4 \times 6 = 24$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

Task This task is in four parts. Consider

$$\Delta = \begin{vmatrix} 1 & 4 & 8 & 2 \\ 2 & -1 & 1 & -3 \\ 0 & 2 & 4 & 2 \\ 0 & 3 & 6 & 3 \end{vmatrix}$$

(a) Use property 2 to find another matrix whose determinant is equal to Δ:

Your solution

Answer

$\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 4 & -1 & 2 & 3 \\ 8 & 1 & 4 & 6 \\ 2 & -3 & 2 & 3 \end{vmatrix}$, by transposing the matrix.

(b) Now expand along the top row to express Δ as the sum of two products, each of a number and a 3×3 determinant:

Your solution

Answer

$$\Delta = 1 \times \begin{vmatrix} -1 & 2 & 3 \\ 1 & 4 & 6 \\ -3 & 2 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ 2 & 2 & 3 \end{vmatrix}$$

(c) Use the statement after property 4 to show that the second of the 3×3 determinants is zero:

Your solution

Answer

In the second 3×3 determinant, row $2 = 2 \times$row 1 hence the determinant has value zero.

(d) Use the statement after property 4 to evaluate the first determinant:

Your solution

Answer

In the first 3×3 determinant column $3 = \frac{3}{2} \times$ column 2. Hence this determinant is also zero. Therefore $\Delta = 0$.

Exercises

1. Use Laplace expansion along the 1st row to determine

$$\begin{vmatrix} 3 & 1 & -4 \\ 6 & 9 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

Show that the same value is obtained if you choose any other row or column for your expansion.

2. Using any of the properties of determinants to minimise the arithmetic, evaluate

(a) $\begin{vmatrix} 12 & 27 & 12 \\ 28 & 18 & 24 \\ 70 & 15 & 40 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 4 & 6 & 4 \\ 0 & 4 & 6 & 9 \\ 2 & 1 & 4 & 0 \\ 1 & 2 & 3 & 2 \end{vmatrix}$

3. Find the cofactors of x, y, z in the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ x & y & z \end{vmatrix}$$

4. Prove that, no matter what the values of $x, y, z,$ are

$$\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Answers

1. $3\begin{vmatrix} 9 & -2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 6 & -2 \\ -1 & 1 \end{vmatrix} - 4\begin{vmatrix} 6 & 9 \\ -1 & 2 \end{vmatrix} = 3(9+4) - 1(6-2) - 4(12+9) = -49$

2. (a) Take out common factors in rows and columns

$720\begin{vmatrix} 2 & 3 & 1 \\ 7 & 3 & 3 \\ 7 & 1 & 2 \end{vmatrix} = 720\begin{vmatrix} 0 & 0 & 1 \\ 1 & -6 & 3 \\ 3 & -5 & 2 \end{vmatrix}$ using $(-2C_3 + C_1)$ then $(-3C_3 + C_2)$.

The value of the determinant (expand along top row) is then easily found as $720 \times 13 = 9360$.

(b) Zero since (row 1) is $2 \times$ (row 4).

3. Cofactors of x, y, z are $1, -2, 1$ respectively.

The Inverse of a Matrix 7.4

Introduction

In number arithmetic every number a $(\neq 0)$ has a reciprocal b written as a^{-1} or $\dfrac{1}{a}$ such that $ba = ab = 1$. Some, but not all, square matrices have inverses. If a square matrix A has an inverse, A^{-1}, then

$$AA^{-1} = A^{-1}A = I.$$

We develop a rule for finding the inverse of a 2×2 matrix (where it exists) and we look at two methods of finding the inverse of a 3×3 matrix (where it exists).

Non-square matrices do not possess inverses so this Section only refers to square matrices.

Prerequisites

Before starting this Section you should ...

- be familiar with the algebra of matrices
- be able to calculate a determinant
- know what a cofactor is

Learning Outcomes

On completion you should be able to ...

- state the condition for the existence of an inverse matrix
- use the formula for finding the inverse of a 2×2 matrix
- find the inverse of a 3×3 matrix using row operations and using the determinant method

1. The inverse of a square matrix

We know that any non-zero number k has an inverse; for example 2 has an inverse $\frac{1}{2}$ or 2^{-1}. The inverse of the number k is usually written $\frac{1}{k}$ or, more formally, by k^{-1}. This numerical inverse has the property that

$$k \times k^{-1} = k^{-1} \times k = 1$$

We now show that an inverse of a matrix can, in certain circumstances, also be defined.

Given an $n \times n$ square matrix A, then an $n \times n$ square matrix B is said to be the **inverse matrix** of A if

$$AB = BA = I$$

where I is, as usual, the identity matrix (or unit matrix) of the appropriate size.

Example 6

Show that the inverse matrix of $A = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$ is $B = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

Solution

All we need do is to check that $AB = BA = I$.

$$AB = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reader should check that $BA = I$ also.

We make three important remarks:

- Non-square matrices do not have inverses.
- The inverse of A is usually written A^{-1}.
- Not all square matrices have inverses.

Task

Consider $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, and let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a possible inverse of A.

(a) Find AB and BA:

Your solution

$AB =$ $\qquad\qquad$ $BA =$

Answer

$AB = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}, \qquad BA = \begin{bmatrix} a+2b & 0 \\ c+2d & 0 \end{bmatrix}$

(b) Equate the elements of AB to those of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and solve the resulting equations:

Your solution

Answer

$a = 1, \quad b = 0, \quad 2a = 0, \quad 2b = 1$. Hence $a = 1, \quad b = 0, \quad a = 0, \quad b = \frac{1}{2}$. This is not possible!

Hence, we have a contradiction. The matrix A therefore has no inverse and is said to be a **singular matrix**. A matrix which has an inverse is said to be **non-singular**.

- If a matrix has an inverse then that inverse is unique.
  Suppose B and C are both inverses of A. Then, by definition of the inverse,

  $AB = BA = I$ **and** $AC = CA = I$

Consider the two ways of forming the product CAB

1. $CAB = C(AB) = CI = C$
2. $CAB = (CA)B = IB = B$.

Hence $B = C$ and the **inverse is unique**.

- There is no such operation as division in matrix algebra.

  We do not write $\dfrac{B}{A}$ but rather

  $A^{-1}B$ or BA^{-1},

  depending on the order required.

- Assuming that the square matrix A has an inverse A^{-1} then the solution of

  the system of equations $AX = B$ is found by pre-multiplying both sides by A^{-1}.

$$\begin{array}{lrcll} & AX & = & B & \\ \text{pre-multiplying by } A^{-1}: & A^{-1}(AX) & = & A^{-1}B, & \\ \text{using associativity:} & A^{-1}A)X & = & A^{-1}B & \\ \text{using } A^{-1}A = I: & IX & = & A^{-1}B, & \\ \text{using property of } I: & X & = & A^{-1}B & \text{which is the solution we seek.} \end{array}$$

2. The inverse of a 2×2 matrix

In this subsection we show how the inverse of a 2×2 matrix can be obtained (if it exists).

Task

Form the matrix products AB and BA where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Your solution

$AB =$ $BA =$

Answer

$$AB = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad-bc)I$$

$$BA = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)I$$

You will see that had we chosen $C = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ instead of B then both products AC and CA will be equal to I. This requires $ad - bc \neq 0$. Hence this matrix C is the inverse of A. However, note, that if $ad-bc = 0$ then A has **no inverse**. (Note that for the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, which occurred in the last task, $ad - bc = 1 \times 0 - 0 \times 2 = 0$ confirming, as we found, that A has no inverse.)

Key Point 8

The Inverse of a 2×2 Matrix

If $ad - bc \neq 0$ then the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a (unique) inverse given by

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that $ad - bc = |A|$, the determinant of the matrix A.

In words: To find the inverse of a 2×2 matrix A we interchange the diagonal elements, change the sign of the other two elements, and then divide by the determinant of A.

Task

Which of the following matrices has an inverse?

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Your solution

Answer

$|A| = 1 \times 3 - 0 \times 2 = 3; \quad |B| = 1 + 1 = 2; \quad |C| = 2 - 2 = 0; \quad |D| = 1 - 0 = 1.$

Therefore, $A,\ B$ and D each has an inverse. C does not because it has a zero determinant.

Task

Find the inverses of the matrices $A,\ B$ and D in the previous Task.

Use Key Point 8:

Your solution

$A^{-1} =$ $\qquad$ $B^{-1} =$ $\qquad$ $C^{-1} =$

Answer

$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}, \ B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \ D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D$

It can be shown that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ represents an **anti-clockwise** rotation through an angle θ in an xy-plane about the origin. The matrix B represents a rotation **clockwise** through an angle θ. It is given therefore by

$$B = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Task

Form the products AB and BA for these 'rotation matrices'. Confirm that B is the inverse matrix of A.

Your solution

$AB =$

$BA =$

Answer

$$\begin{aligned} AB &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Similarly, $BA = I$

Effectively: a rotation through an angle θ followed by a rotation through angle $-\theta$ is equivalent to zero rotation.

3. The inverse of a 3×3 matrix - Gauss elimination method

It is true, in general, that if the determinant of a matrix is zero then that matrix has no inverse. If the determinant is non-zero then the matrix has a (unique) inverse. In this Section and the next we look at two ways of finding the inverse of a 3×3 matrix; larger matrices can be inverted by the same methods - the process is more tedious and takes longer. The 2×2 case could be handled similarly but as we have seen we have a simple formula to use.

The method we now describe for finding the inverse of a matrix has many similarities to a technique used to obtain solutions of simultaneous equations. This method involves operating on the rows of a matrix in order to reduce it to a unit matrix.
The row operations we shall use are

(i) interchanging two rows
(ii) multiplying a row by a constant factor
(iii) adding a multiple of one row to another.

Note that in (ii) and (iii) the multiple could be negative or fractional, or both.

The Gauss elimination method is outlined in the following Key Point:

Key Point 9

Matrix Inverse – Gauss Elimination Method

We use the result, quoted without proof, that:

if a sequence of row operations applied to a square matrix A reduces it to the identity matrix I of the same size then the **same** sequence of operations applied to I reduces it to A^{-1}.

Three points to note:

- If it is impossible to reduce A to I then A^{-1} does not exist. This will become evident by the appearance of a row of zeros.
- There is no unique procedure for reducing A to I and it is experience which leads to selection of the optimum route.
- It is more efficient to do the two reductions, A to I and I to A^{-1}, simultaneously.

Suppose we wish to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix}$$

We first place A and I adjacent to each other.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Phase 1

We now proceed by changing the columns of A *left to right* to reduce A to the form $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$ where $*$ can be any number. This form is called **upper triangular**.

First we subtract row 1 from row 2 and twice row 1 from row 3. 'Row' refers to both matrices.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \\ R2 - R1 \\ R3 - 2R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Now we subtract row 2 from row 3

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \\ \\ R3 - R2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Phase 2

This consists of continuing the row operations to reduce the elements above the leading diagonal to zero.

We proceed *right to left*. We subtract 3 times row 3 from row 1 (the elements in row 2 column 3 is already zero.)

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{matrix} \\ \\ R1 - 3R3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Finally we subtract 3 times row 2 from row 1.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \begin{matrix} \\ \\ R1 - 3R2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Then we have $A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

(This can be verified by showing that $AA^{-1} = I$ or $A^{-1}A = I$.)

Task

Consider $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Use the Gauss elimination method to obtain A^{-1}.

First interchange rows 1 and 2, then carry out the operation (row $3) + \frac{1}{2}$(row 1):

Your solution

Answer

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R1 \leftrightarrow R2 \qquad \Rightarrow \quad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R3 + \tfrac{1}{2}R1 \qquad \Rightarrow \quad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & \frac{7}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Now carry out the operation (row $3) - \frac{7}{2}$(row 2) followed by (row $1) - \frac{1}{3}$(row 3)
and (row $2) + \frac{1}{3}$(row 3):

Your solution

Answer

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & \frac{7}{2} & \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \quad R3 - \tfrac{7}{2}R2 \quad \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad \begin{matrix} R1 - \frac{1}{3}R3 \\ R2 + \frac{1}{3}R3 \end{matrix} \quad \Rightarrow \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} +\frac{7}{6} & +\frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

Next, subtract 3 times row 2 from row 1, then, divide row 1 by 2 and row 3 by (-3).
Finally identify A^{-1}:

Your solution

Answer

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} \frac{7}{6} & \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad R1 - 3R2 \quad \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{10}{6} & \frac{2}{6} & -\frac{4}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \begin{bmatrix} \frac{10}{6} & \frac{2}{6} & -\frac{4}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad \begin{matrix} R1 \div 2 \\ \\ R3 \div (-3) \end{matrix} \quad \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{7}{6} & -\frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{7}{6} & -\frac{1}{6} & -\frac{1}{3} \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$

4. The inverse of a 3×3 matrix - determinant method

This method which employs determinants, is of importance from a theoretical perspective. The numerical computations involved are too heavy for matrices of higher order than 3×3 and in such cases the Gauss elimination approach is prefered.

To obtain A^{-1} using the determinant approach the steps in the following keypoint are followed:

Key Point 10

Matrix Inverse – the Determinant Method

Given a square matrix A:

- Find $|A|$. If $|A| = 0$ then A^{-1} does not exist. If $|A| \neq 0$ we can proceed to find the inverse matrix, as follows.
- Replace each element of A by its cofactor (see Section 7.3).
- Transpose the result to form the **adjoint matrix**, denoted by adj(A)
- Then calculate $A^{-1} = \dfrac{1}{|A|}$ adj(A).

Task

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$. This will require five stages.

(a) First find $|A|$:

Your solution

Answer

$|A| = 0 \times 5 + 1 \times (-1) + 1 \times 7 = 6$

(b) Now replace each element of A by its minor:

Your solution

Answer

$$\begin{bmatrix} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 1 & 7 \\ -1 & 1 & 1 \\ -4 & -2 & -2 \end{bmatrix}$$

(c) Now attach the signs from the array

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

(so that where a $+$ sign is met no action is taken and where a $-$ sign is met the sign is changed) to obtain the matrix of cofactors:

Your solution

Answer

$$\begin{bmatrix} 5 & -1 & 7 \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{bmatrix}$$

(d) Then transpose the result to obtain the adjoint matrix:

Your solution

Answer

Transposing, $\text{adj}(A) = \begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$

(e) Finally obtain A^{-1}:

Your solution

Answer

$A^{-1} = \dfrac{1}{\det(A)}\text{adj}(A) = \dfrac{1}{6}\begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$ as before using Gauss elimination.

Exercises

1. Find the inverses of the following matrices

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

2. Use the determinant method and also the Gauss elimination method to find the inverse of the following matrices

(a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 1 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Answers

1. (a) $-\dfrac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ (c) $\dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

2. (a) $A^{-1} = -\dfrac{1}{2}\begin{bmatrix} 0 & -2 & 1 \\ -2 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}^T = -\dfrac{1}{2}\begin{bmatrix} 0 & -2 & 0 \\ -2 & 4 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

(b) $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$